LIMITS OF GENERALIZED PERIODIC D-SPLINES

BY

DAVID L. RAGOZIN[†]

Department of Mathematics, GN-50, University of Washington, Seattle, WA 98195, USA

ABSTRACT

Results of Schoenberg and others on limits of periodic splines as their order, m, goes to infinity are extended to sequences of D^m-splines determined by the powers of an unbounded non-negative self-adjoint operator D on a Hilbert space, **H**, and an evaluation map **L** from **H** to $\mathbf{R}^n$. All such limits lie in the lowest frequency n-dimensional invariant subspace for D, $\mathbf{T}_n^*$. When each term in the sequence is the D^m-spline whose image under **L** matches a fixed vector, **y**, (an L-interpolant), then the limit is the L-interpolant to **y** from $\mathbf{T}_n^*$. When the terms are *smoothing* splines derived from **y** then the limit exists when the smoothing parameter goes to 0 as t^{-m}. If t is not an eigenvalue, α_i, of D, the limit is the L-least squares best fit to **y** from $\mathbf{T}_l^*$, $l = \operatorname{card}\{j : \alpha_j < t\}$.

1. Introduction

This paper provides an operator theoretic version of some results of Schoenberg, Golitschek, Cavaretta and Newman, and Ragozin [4, 2, 1, 3], on the limiting behavior of interpolating or smoothing splines as the degree tends to infinity. Our basic setting consists of:

(1.1)

(i) An abstract (real) Hilbert space, **H**.

(ii) A non-negative unbounded self-adjoint operator D on **H** with *finite dimensional* spectral projections.

(iii) An unbounded linear map **L** from **H** *onto* $\mathbf{R}^n$ with component functionals $l_1, l_2, \ldots, l_n$.

[†] Research supported in part by NSF grant MCS-8308349.
Received November 21, 1983

Associated with $\mathbf{H}$, D, $\mathbf{L}$, are the family of Hilbert spaces

$$\mathbf{H}^m := \mathrm{Dom}(\mathrm{D}^{m/2}), \textit{ with semi-norm } |h|_m := \|D^{m/2}h\| \tag{1.2}$$
$$(= \langle h, D^m h\rangle^{1/2}, h \in \mathrm{Dom}(D^m)).$$

We assume that there exists an m_0 such that $\mathbf{L}$ is defined on $\mathbf{H}^m$, for $m \geqq m_0$, and is continuous with respect to the norm

$$\|h\|_m := (|h|_m^2 + \|h\|^2)^{1/2}$$

on $\mathbf{H}^m$. In everything that follows we shall always assume $m \geqq m_0$. From 1.1(ii) it follows that the *eigenvalues* of D form an *increasing* sequence. We let

$$\mathrm{Spec}(\mathrm{D}) = \{\alpha_1 \leqq \alpha_2 \leqq \cdots\}, \tag{1.3}$$

and we also assume that $\mathbf{L}$ continues to have maximal rank when restricted to any of the D-invariant spaces

$$\mathbf{T}_k := \left\{ h \in \mathbf{H} : h = \sum_{i=1}^{k} \beta_i h_i, \mathrm{D}h_i = \alpha_i h_i \right\} \tag{1.4}$$

of D-*polynomials of degree at most* k. Moreover, we assume

$$\dim \ker(\mathrm{D}) \leqq n. \tag{1.5}$$

Our main object is to determine the limiting behavior of certain sequences of $2m$*'th order* DL-*splines* as m tends to infinity. The space of $2m$*'th order* DL-*splines* is defined by

$$S^m(\mathrm{D},\mathbf{L}) := \left\{ h \in \mathrm{Dom}(\mathrm{D}^{m/2}) : \textit{for some } \mathbf{y} \textit{ in } \mathbf{R}^n, h = \underset{\mathbf{L}g=\mathbf{y}}{\operatorname{argmin}} |g|_m^2 \right\}. \tag{1.6}$$

It follows from our assumptions that $\mathbf{L}$ restricted to $S^m(\mathrm{D},\mathbf{L})$ is invertible (see 2.2), so we can define the $2m$*'th order spline interpolant* to $\mathbf{y}$ in $\mathbf{R}^n$, denoted $\mathrm{S}_{0,m}\mathbf{y}$, by

$$\mathrm{S}_{0,m}\mathbf{y} = h \in S^m(\mathrm{D},\mathbf{L}), \qquad \textit{if and only if } \mathbf{L}h = \mathbf{y} \tag{1.7}$$
$$\left(\textit{if and only if } h = \underset{\mathbf{L}g=\mathbf{y}}{\operatorname{argmin}} |g|_m^2\right).$$

We intend to show that $\lim_{m\to\infty} \mathrm{S}_{0,m}\mathbf{y}$ always exists and to characterize this limit as an L-interpolant to $\mathbf{y}$ from the n'th degree D-polynomials, $\mathbf{T}_n$, i.e. a $t \in \mathbf{T}_n$ with $\mathbf{L}t = \mathbf{y}$. If $\alpha_n = \alpha_{n+1}$, so $\dim \mathbf{T}_n > n$, there are many interpolants. To obtain a precise value for $\lim_{m\to\infty} \mathrm{S}_{0,m}\mathbf{y}$ we must restrict $\mathbf{T}_n$ to the n-dimensional subspace, $\mathbf{T}_n^*$, whose members are characterized by the fact that their components belonging to the eigenvalue α_n are orthogonal to $\ker(\mathbf{L})$.

More generally, we shall examine the limits as $m \to \infty$ of the smoothing DL-spline sequences $S_{\lambda_m,m}\mathbf{y}$, where $S_{\lambda,m}\mathbf{y}$ is defined by

$$(1.8) \qquad S_{\lambda,m}\mathbf{y} = \underset{g \in \mathbf{H}^m}{\operatorname{argmin}} \sum_{i=1}^{n} (y_i - l_i(g))^2 + \lambda \| g \|_m^2 .$$

Under the assumptions we have made it will follow that $S_{\lambda,m}\mathbf{y} \in S^m(\mathrm{D},\mathbf{L})$. We shall show that $\lim_{m\to\infty} S_{\lambda_m,m}\mathbf{y}$ exists for all $\mathbf{y} \in \mathbf{R}^n$ if and only if $\lim_{m\to\infty} \lambda_m \alpha_j^{2m} = d(j)$ exists in the extended half-line $[0,\infty]$ for all integral j with $1 \leqq j \leqq n$. Moreover, when $l = \max\{j \leqq n : d(j) < \infty\}$, $\lim_{m\to\infty} S_{\lambda_m,m}\mathbf{y}$ is a D-polynomial of degree l, which can be characterized, when $d(l) = 0$, as the $\mathbf{L}$-least squares best fit to $\mathbf{y}$ from $\mathbf{T}_l$, i.e. the $t \in \mathbf{T}_l$ which minimizes $\sum_{i=1}^{n} (y_i - l_i(t))^2$.

A number of settings in which our assumptions hold are easily described. The simplest is $\mathbf{H} = L_2^{\circ}([0,1])$, the 1-periodic L_2 functions, with $\mathrm{D} = -d^2/dt^2$ and $\mathbf{L}f = [f(x_i)]$, the evaluation mapping on the set $\Delta = \{0 < x_1 < \cdots < x_n \leqq 1\}$. For this example what follows just recovers the work in [3]. $\mathbf{H}^m$ is the standard periodic Sobolev space, and the $2m$'th order DL-splines are just the usual periodic polynomial splines of order $2m$. This example has dictated our choice of nomenclature for the general case. Other choices for $\mathbf{L}$ lead to more exotic spaces of periodic splines.

The simplest generalizations of the preceding example are the multipli-periodic splines on $\mathbf{R}^k$. These arise when $\mathbf{H} = L_2^{\circ}([0,1]^k)$, the multipli-periodic functions on $[0,1]^k$ with $\mathrm{D} = -\sum_{i=1}^{k} \partial^2/\partial t_i^2$ and $\mathbf{L}f = [f(\mathbf{x}_i)]$ the evaluation map on a set $\Delta = \{\mathbf{x}_j : j = 1,\ldots,n\}$. The points in Δ must be restricted by some general position requirements for the maximal rank assumption, 1.4, to hold. Again the spaces $\mathbf{H}^m$ are periodic Sobolev spaces, and the $\mathbf{T}_k$ are spaces of trigonometric polynomials.

A vast collection of generalizations are provided by letting $\mathbf{H}$ be the L_2 space of any compact Riemannian manifold, M, without boundary, with D the negative of the Laplace–Beltrami operator on M. $\mathbf{L}$ can be an evaluation map, provided the points are in general position with respect to the first n eigenfunctions for D. One simple case is when M is the k-sphere, S^k. Then the spaces $\mathbf{T}_k$ are spaces of generalized spherical harmonics, but the $2m$'th order DL-splines are difficult to describe explicitly in this case, since they involve fundamental solutions for D^m which cannot be given in a simple closed form. (See [5].)

2. Interpolating and smoothing DL-splines

Our development requires a few basic facts about DL-splines. We need to show that our assumptions on D, $\mathbf{L}$, and $\mathbf{T}_n$ are enough to recover most of the standard facts about interpolating and smoothing splines.

Our first goal is to show that interpolating DL-splines always exist and are unique. A useful result toward this goal is

LEMMA 2.1. *Given* $\mathbf{y}$ *in* $\mathbf{H}$, *if* $h = \operatorname{argmin}_{\mathbf{L}g=\mathbf{y}} |g|_m^2$, *then* $\mathrm{D}^{m/2}h \perp \ker(\mathbf{L}) \cap \mathbf{H}^m$.

PROOF. From the minimization property of h it follows that $|h + tg|_m^2 \geqq |h|_m^2$ for all $t \in \mathbf{R}$ if $\mathbf{L}g = 0$. So standard Hilbert space minimization arguments imply $\langle \mathrm{D}^{m/2}h, \mathrm{D}^{m/2}g \rangle = 0$ for all such g. ■

Now we can prove

THEOREM 2.2. *Suppose the Hilbert space* $\mathbf{H}$ *and the (unbounded) operators* $\mathrm{D}, \mathbf{L}$ *satisfy the assumptions in section* 1, *in particular* (1.5). *Then* $\mathbf{L}: S^m(\mathrm{D}, \mathbf{L}) \rightarrow \mathbf{R}^n$ *is invertible, i.e.*

(2.3) *For all* $\mathbf{y}$ *in* $\mathbf{R}^n$, *there exists a unique* $h \in \mathbf{H}^m$ *which solves* $h = \underset{\mathbf{L}g=\mathbf{y}}{\operatorname{argmin}} |g|_m^2$.

PROOF. One of the assumptions in Section 1, 1.4, was that $\mathbf{L}$ has maximal rank on the D-invariant space $\mathbf{T}_n$. But $\mathbf{T}_n$ is included in $\mathbf{H}^m$, so $\mathbf{L}$ has rank n on $\mathbf{H}^m$. Hence the hyperplane $\{g \in \mathbf{H}^m : \mathbf{L}g = \mathbf{y}\}$ is non-empty. Now standard theory shows that the weakly lower semi-continuous convex function $|g|_m^2$ attains its *infimum* on this hyperplane since it is bounded below.

To show that the minimum is attained at exactly one point, suppose h_1 and h_2 are both minimizers. Then $\mathbf{L}(h_1 - h_2) = 0$ so 2.1 shows $\langle \mathrm{D}^{m/2}h_i, \mathrm{D}^{m/2}(h_1 - h_2) \rangle = 0$. Hence $\langle \mathrm{D}^{m/2}(h_1 - h_2), \mathrm{D}^{m/2}(h_1 - h_2) \rangle = 0$, and thus

$$h_1 - h_2 \in \ker(\mathrm{D}^{m/2}) \cap \ker(\mathbf{L}) = \ker(\mathrm{D}) \cap \ker(\mathbf{L}).$$

But either $\ker(\mathrm{D}) = \{\mathbf{0}\}$ or 0 is the smallest eigenvalue of D and $\ker(\mathrm{D}) = \mathbf{T}_1$. In the first case $\ker(\mathrm{D}) \cap \ker(\mathbf{L}) = \{0\}$, while in the second case the maximal rank assumption on $\mathbf{L}$ also implies that intersection is zero since $\dim \ker(\mathrm{D}) \leqq n$ by 1.5. Hence $h_1 = h_2$ holds in either case. ■

COROLLARY 2.4. $S^m(\mathrm{D}, \mathbf{L}) = \ker(\mathbf{L})^{\perp_m}$ *where* $\perp_m$ *means the orthogonal complement with respect to the semi-inner product* $\langle \mathrm{D}^{m/2}h, \mathrm{D}^{m/2}g \rangle$ *on* $\mathbf{H}^m$.

PROOF. The inclusion $S^m(\mathrm{D}, \mathbf{L}) \subseteq \ker(\mathbf{L})^{\perp_m}$ is just 2.1. In the opposite direction if $h \in \ker(\mathbf{L})^{\perp_m}$ then the existence of the DL-spline interpolant $\mathrm{S}_{0,m}\mathbf{L}h$ and 2.1 imply $h - \mathrm{S}_{0,m}\mathbf{L}h \in \ker(\mathbf{L})^{\perp_m} \cap \ker(\mathbf{L})$. So $h - \mathrm{S}_{0,m}\mathbf{L}h \in \ker(\mathrm{D}^{m/2}) \subseteq S^m(\mathrm{D}, \mathbf{L})$. From this the containment $S^m(\mathrm{D}, \mathbf{L}) \supseteq \ker(\mathbf{L})^{\perp_m}$ follows. ■

Our second goal is to show that the minimization problem in 1.8 has a unique solution which is a DL-spline.

PROPOSITION 2.5. *For any $\lambda > 0$ and any $\mathbf{y} \in \mathbf{R}^n$ there exists a unique $h \in \mathbf{H}^m$ with*

$$h = \operatorname*{argmin}_{g \in \mathbf{H}^m} \sum_{i=1}^{n} (y_i - l_i(g))^2 + \lambda \, | g |_m^2. \tag{2.6}$$

That h is in $S^m(\mathrm{D}, \mathbf{L})$.

PROOF. The existence of solutions to the minimization problem inherent in 2.6 follows just as in 2.3 since the function being minimized is lower semi-continuous, convex, and bounded below. Moreover, any solution, h, to this quadratic minimization problem must satisfy

$$0 = \sum_{i=1}^{n} (y_i - l_i(h)) l_i(g) - \lambda \langle \mathrm{D}^{m/2} h, \mathrm{D}^{m/2} g \rangle, \qquad \textit{all } g \in \mathbf{H}^m$$

by standard orthogonality arguments. When g is restricted to ker(**L**) the first sum is zero and this equation implies $h \in \ker(\mathbf{L})^{\perp_m}$. Hence any solution is in $S^m(\mathrm{D}, \mathbf{L})$ by 2.4.

The uniqueness of the solution h can be seen in the following way. Both summands in the expression being minimized are convex in g and the first summand is strictly convex as a function of $\mathbf{L}g$. Hence any two solutions would have the same values for $\mathbf{L}g$. But since they would both be DL-splines, they must be the same by the uniqueness of interpolating DL-splines, 2.3. ∎

3. Limit theorems for interpolating and smoothing DL-splines

This section contains the statement and proofs of our main results. We closely parallel the proofs for periodic splines [3] and begin by showing that any limit of $\mathrm{S}_{\lambda,m}\mathbf{y}$ must be in the space $\mathbf{T}_n$ of D-polynomials of degree n. When $\alpha_n = \alpha_{n+1}$ the highest frequency term of this limit must have a special form. These results allow us to reduce our work to a question about finite dimensional spaces, whose resolution leads to the main theorem.

We assume that the data vector $\mathbf{y}$ is fixed and that the splines $\mathrm{S}_{\lambda_m,m}\mathbf{y}$ are defined by 1.7 if $\lambda_m = 0$ or by 1.8 otherwise. Let us decompose $\mathrm{S}_{\lambda_m,m}\mathbf{y}$ according to the eigenbasis for D as

$$\mathrm{S}_{\lambda_m,m}\mathbf{y} = t_m + r_m, \qquad t_m \in \mathbf{T}_n, \quad r_m \in \mathbf{T}_n^{\perp}. \tag{3.1}$$

Our first step in studying the convergence of $\mathrm{S}_{\lambda_m,m}\mathbf{y}$ will be to show that $\mathrm{D}^k r_m$ converges to zero, no matter what λ_m's are chosen. The key to this is the fact that if s_0 is any $\mathbf{T}_n$ interpolant for $\mathbf{y}$ (of course such s_0 exist by the rank assumption at 1.4), then

$$| \mathrm{S}_{\lambda_m,m}\mathbf{y} |_m^2 \leq | s_0 |_m^2. \tag{3.2}$$

This follows from the minimizing property of the DL-spline interpolant if $\lambda_m = 0$ or, when $\lambda_m > 0$, from the fact that the minimization property of $S_{\lambda_m,m}\mathbf{y}$ in 1.8 shows

$$\lambda \,|\, S_{\lambda_m,m}\mathbf{y}\,|_m^2 \leqq \sum_{k=1}^{n} |\, y_k - l_k\,(S_{\lambda_m,m}\mathbf{y})\,|^2 + \lambda \,|\, S_{\lambda_m,m}\mathbf{y}\,|_m^2$$

$$\leqq \Sigma \,|\, y_k - l_k\,(s_0)\,|^2 + \lambda \,|\, s_0\,|_m^2 \tag{3.3}$$

$$= \lambda \,|\, s_0\,|_m^2.$$

Now the desired convergence of r_m is a consequence of

PROPOSITION 3.4. *Suppose g_m is any sequence with g_m in $\mathbf{H}^m$ and suppose there exists s_0 in $\mathbf{T}_n$ with $|\, g_m\,|_m^2 \leqq |\, s_0\,|_m^2$ for all m. If $g_m = t_m + r_m$, as in* 3.1, *then for each k, $|\, r_m\,|_k = \langle r_m, D^k r_m\rangle^{1/2} \to 0$ as $m \to \infty$. Hence any limit point of $\{g_m\}$ lies in $\mathbf{T}_n$.*

PROOF. First note that the remainder terms, r_m, are in $\mathbf{H}^m$ since $r_m = g_m - t_m$ and the D-polynomial t_m is in $\mathbf{H}^m$. Hence if we let

$$\alpha_{n+} = \min\{\alpha_i : \alpha_i > \alpha_n\} \tag{3.5}$$

then the D-eigenbasis expansion of g_m leads to

$$\langle r_m, \mathrm{D}^k r_m\rangle \leqq \langle r_m, \mathrm{D}^{k-m}\mathrm{D}^m r_m\rangle \leqq \alpha_{n+}^{k-m}\langle r_m, \mathrm{D}^m r_m\rangle$$

$$\leqq \alpha_{n+}^{k-m}\langle g_m, \mathrm{D}^m g_m\rangle \leqq \alpha_{n+}^{k-m}\langle s_0, \mathrm{D}^m s_0\rangle.$$

But s_0 is in $\mathbf{T}_n$ so

$$\langle r_m, \mathrm{D}^k r_m\rangle \leqq \alpha_{n+}^{k-m}\,\alpha_n^m\langle s_0, s_0\rangle,$$

i.e. $|\, r_m\,|_k \to 0$ at the rate $(\alpha_n/\alpha_{n+})^{m/2}$. ■

A slight refinement of this result applied to the sequence $S_{\lambda_m,m}\mathbf{y}$ follows from the fact that then each g_m is a DL-spline. Specifically, if $\alpha_n = \alpha_{n+1}$ then $\dim \mathbf{T}_n > n$, so $\mathbf{L}$ restricted to $\mathbf{T}_n$ has a non-trivial kernel. We let

$$Z_n = \ker(\mathbf{L}) \cap \mathbf{T}_n$$

denote the n'th degree D-polynomials vanishing at each l_i. Each z in Z_n can be written in the form

$$z = z_n + p, \qquad \text{with } Dz_n = \alpha_n z_n, \quad p \text{ lower frequency.} \tag{3.6}$$

("Lower frequency" means any eigenvector occurring in p belongs to an eigenvalue *less than* α_n.) Let the *L-proximal* k'th degree D-polynomials be the set

(3.7) $$\mathbf{T}_k^* = \{t_k \in \mathbf{T}_k : \langle t_k, z_n \rangle = 0, \text{ for all } z_n \text{ as in } 3.6\}.$$

Then any limit of $S_{\lambda_m,m}\mathbf{y}$ lies in $\mathbf{T}_n^*$ as a result of

PROPOSITION 3.8. *For each m, let g_m be a DL-spline and suppose the sequence $\{g_m\}$ satisfies the hypotheses of Proposition 3.4. Then*

$$\lim_{m \to \infty} \langle g_m, z_n \rangle = 0, \qquad \text{for any } z_n \text{ as in } (3.6).$$

PROOF. Since $\mathbf{L}z = 0$, for any z in Z_n, 2.4 says $0 = \langle D^{m/2} g_m, D^{m/2} z \rangle$. So, from 3.6, we have

$$0 = \langle g_m, D^m z_n \rangle + \langle g_m, D^m p \rangle = \alpha_n^m \langle g_m, z_n \rangle + \langle g_m, D^m p \rangle.$$

Now since p has lower frequency than z_n, it involves eigenvectors associated with eigenvalues which are no larger than

$$\alpha_{n-} = \max\{\alpha_i : \alpha_i < \alpha_n\}.$$

Thus the bound $|g_m|_m^2 \leqq |s_0|_m^2$, the fact that $s_0 \in \mathbf{T}_n$, and the Cauchy–Schwarz inequality can be applied to obtain

$$\begin{aligned} \alpha_n^m |\langle g_m, z_n \rangle| = \langle g_m, D^m p \rangle &\leqq |g_m|_m |p|_m \\ &\leqq |s_0|_m \alpha_{n-}^{m/2} \langle p, p \rangle^{1/2} \leqq \alpha_n^{m/2} \|s_0\| \alpha_{n-}^{m/2} \|p\|. \end{aligned}$$

So

$$|\langle g_m, z_n \rangle| \leqq (\alpha_{n-}/\alpha_n)^{m/2} \|s_0\| \|p\|.$$

and $\langle g_m, z_n \rangle \to 0$ at a rate less than or equal to $(\alpha_{n-}/\alpha_n)^{m/2}$. ■

Now we can examine the convergence properties of $S_{\lambda_m,m}\mathbf{y}$ as m goes to infinity. From the preceding two propositions any limit point of this sequence of splines must belong to the space $\mathbf{T}_n^* \subset \mathbf{T}_n$. Since $\mathbf{T}_n^*$ is D-invariant, we can find a D-eigenbasis $\{e_j\}$ for $\mathbf{T}_n^*$. Our assumptions imply that we can order the $\{e_j\}$ so that $De_j = \alpha_j e_j$. (Note that the subset $\mathbf{T}_n^* \subset \mathbf{T}_n$ is defined via conditions which only affect the eigenfunctions associated with α_n.) We shall refer to this basis as the *L-proximal Fourier basis* and we define the *L-proximal Fourier coefficients* of $S_{\lambda,m}\mathbf{y}$ by

$$c_j(\lambda, m) = \langle S_{\lambda,m}\mathbf{y}, e_j \rangle, \qquad 1 \leqq j \leqq n.$$

Let

$$R_m = S_{\lambda,m}\mathbf{y} - \sum_{j=1}^{n} c_j(\lambda, m)e_j$$

be the modified remainder. Then 3.2, 3.4 and 3.8 imply that $\lim_{m\to\infty} D^k R_m = 0$, for all k. Thus any convergence of $S_{\lambda_m,m}\mathbf{y}$ will be completely determined by the convergence of the coefficients $c_j(\lambda_m, m)$.

A useful expression relating these coefficients for $S_{\lambda_m,m}\mathbf{y}$ and the data $\mathbf{y}$ can be derived from the orthogonality conditions which follow from the minimization property 1.8 used to define $S_{\lambda,m}\mathbf{y}$. Specifically we must have

$$(3.9)\qquad \begin{aligned}\sum_{k=1}^{n} l_k(e_j)(y_k - l_k(S_{\lambda,m}\mathbf{y})) &= \lambda\langle D^{m/2}e_j, D^{m/2}S_{\lambda,m}\mathbf{y}\rangle \\ &= \lambda\langle D^m e_j, S_{\lambda,m}\mathbf{y}\rangle = \lambda\alpha_j^m c_j(\lambda, m).\end{aligned}$$

Note that this equation also holds if $\lambda = 0$.

We can give this last equation a particularly useful expression if we let $\{f_i\}$ be the basis for $\mathbf{T}_n^*$ dual to $\{e_j\}$ with respect to the discrete inner product

$$\langle f, g\rangle_L = \sum_{k=1}^{n} l_k(f)l_k(g).$$

This is an inner-product on $\mathbf{T}_n^*$, since $\ker(\mathbf{L})\cap\mathbf{T}_n^* = \{0\}$, which follows from 1.8 and the definition 3.7 of $\mathbf{T}_n^*$. Now the coefficients have the representation

$$c_j(\lambda, m) = \left\langle f_j, \sum_{i=1}^{n} c_i(\lambda, m)e_i\right\rangle_L = \langle f_j, S_{\lambda,m}\mathbf{y} - R_m\rangle_L.$$

So 3.9 can be rewritten as

$$(3.10)\qquad \langle \mathbf{L}e_j, \mathbf{y}\rangle_{\mathbf{R}^n} = \langle e_j, S_{\lambda,m}\mathbf{y}\rangle_L + \lambda\alpha_j^m\langle f_j, S_{\lambda,m}\mathbf{y} - R_m\rangle_L.$$

If we introduce the matrices

$$E = [l_i(e_j)],\quad F = [l_i(f_j)],\quad D_m(\lambda) = \mathrm{diag}(\lambda\alpha_j^m)$$

then we can summarize the equations in 3.10 as

$$(3.11)\qquad E^*\mathbf{y} = E^*\mathbf{L}S_{\lambda,m}\mathbf{y} + D_m(\lambda)F^*\mathbf{L}S_{\lambda,m}\mathbf{y} - D_m(\lambda)F^*\mathbf{L}R_m.$$

Since $\{f_j\}$ is dual to $\{e_j\}$, $F = (E^*)^{-1}$, and an alternative summary is

$$\mathbf{y} = (I + FD_m(\lambda)F^*)\mathbf{L}S_{\lambda,m}\mathbf{y} - FD_m(\lambda)F^*\mathbf{L}R_m.$$

Or "solving" for $\mathbf{L}S_{\lambda,m}\mathbf{y}$

$$(3.12)\qquad \mathbf{L}S_{\lambda,m}\mathbf{y} = (I + FD_m(\lambda)F^*)^{-1}\mathbf{y} + (I + FD_m(\lambda)F^*)^{-1}FD_m(\lambda)F^*(\mathbf{L}R_m).$$

From these relationships we can see that the limiting behavior of $S_{\lambda_m,m}\mathbf{y}$ is related to the limiting behavior of the positive definite matrices $I + FD_m(\lambda)F^*$ and their inverses. Specifically we have

LEMMA 3.13. *The limit as $m \to \infty$ of $S_{\lambda_m,m}\mathbf{y}$ exists for each $\mathbf{y}$ if and only if $\lim_{m\to\infty}(I + FD_m(\lambda_m)F^*)^{-1}$ exists.*

PROOF. Just as in 2.13 in [3]. The key ideas are that $S_{\lambda_m,m}\mathbf{y}$ converges if and only if $\mathbf{L}S_{\lambda_m,m}\mathbf{y}$ converges and the coefficient of $\mathbf{L}R_m$ in 3.12 is a positive definite matrix which is less than I. ■

Now we recall Proposition 2.14 in [3].

PROPOSITION 3.14. *If E^*E is any positive definite matrix and $D_m = \operatorname{diag}(d_{m,jj})$ is a sequence of non-negative diagonal matrices, then*

$$\lim_{m\to\infty}(E^*E + D_m)^{-1} \text{ exists if and only if}$$

$$\text{for each } j,\quad \lim_{m\to\infty} d_{m,jj} \text{ exists in } [0,\infty]. \qquad ■$$

When we put all these pieces together we get our main result.

THEOREM 3.15. *A non-negative sequence $\{\lambda_m\}$ has the property that for each n-vector $\mathbf{y}$ the DL-splines $S_{\lambda_m,m}\mathbf{y}$ of order $2m$ based on $\mathbf{y}$ converge as $m \to \infty$ if and only if*

$$\lim_{m\to\infty}\lambda_m\alpha_j^m \text{ exists in } [0,\infty], \quad \text{for all integers } j \text{ with } j \leqq n. \tag{3.16}$$

Whenever λ_m satisfies 3.16, $\lim_{m\to\infty}S_{\lambda_m,m}\mathbf{y} = g_\infty$ is in $\mathbf{T}_l^*$, the L-proximal D-polynomials of degree at most l, where $l = \max\{j : \lim_{m\to\infty}\lambda_m\alpha_j^m < \infty\}$. If $d(l) = \lim_{m\to\infty}\lambda_m\alpha_l^m$ and $g_\infty = \sum_{k=1}^{l} c_k e_k$ is the expansion of the limit with respect to the L-proximal Fourier basis then the coefficients c_k are determined by the equations

$$\sum_{k=1}^{l}\langle \mathbf{L}e_j, \mathbf{L}e_k\rangle c_k = \langle \mathbf{L}e_j, \mathbf{y}\rangle, \qquad \alpha_j < \alpha_l,$$

$$d(l)c_l + \sum_{k=1}^{l}\langle \mathbf{L}e_j, \mathbf{L}e_k\rangle c_k = \langle \mathbf{L}e_j, \mathbf{y}\rangle, \qquad \alpha_j = \alpha_l.$$

PROOF. As in Theorem 2.19 of [3] the $\lim_{m\to\infty}S_{\lambda_m,m}\mathbf{y}$ exists, for all $\mathbf{y}$, if and only if the given limits of $\lambda_m\alpha_j^m$ exist in $[0,\infty]$ as a result of 3.4, 3.8 and 3.13. The fact that the limit is in $\mathbf{T}_l^*$ and the equations for the coefficients, c_k, of the limit follow as in the cited Theorem by passing to the limit in the equations of 3.10, when those with index j, $\alpha_j > \alpha_l$ (i.e. those for which $\lim_{m\to\infty} \alpha_j^m \lambda_m = \infty$) are divided by $\alpha_j^m \lambda_m$. ■

REMARK. Note that when $\lambda_m = 0$ for all m, or, more generally, when $d(l) = 0$, $l = n$, then the equations for the c_k simply say $\langle g_\infty, e_j \rangle_L = \langle \mathbf{y}, e_j \rangle_L$. These determine g_∞ as the (proximal) L-interpolant to $\mathbf{y}$ from $\mathbf{T}_n^*$. When $d(l) = 0$, $l < n$, then these equations are the usual normal equations for the L-least squares best fit to $\mathbf{y}$ from the D-polynomials, $\mathbf{T}_l$.

REFERENCES

1. A. S. Cavaretta, Jr. and D. J. Newman, *Periodic interpolating splines and their limits*, Indag. Math. **40** (1978), 515–526.

2. M. von Golitschek, *On the convergence of interpolating periodic spline functions of high degree*, Numer. Math. **19** (1972), 146–154.

3. D. L. Ragozin, *Limits of periodic smoothing splines*, Indag. Math., to appear, 1983.

4. I. J. Schoenberg, *Notes on spline functions I. The limits of the interpolating periodic spline functions as their degree tends to infinity*, Indag. Math. **34** (1972), 412–422.

5. G. Wahba, *Spline interpolation and smoothing on the sphere*, SIAM J. Sci. Stat. Comput. **2** (1981), 5–16.